Journal of Statistical Physics, Vol. 53, Nos. 3/4, 1988

Universality Two-Dimensional Kauffman Model for Parallel and Sequential Updating

Susanne Stölzle[1,2]

Received June 28, 1988

Kauffman's random Boolean nets are studied on the square lattice by computer simulation comparing parallel and sequential updating of the automata. At the phase transition the fractal dimensions of time and actual damage are found to be independent of the updating method.

KEY WORDS: Kauffman model; cellular automata; parallel and sequential updating.

Almost 20 years ago, the most disordered cellular automaton was introduced by Kauffman[1] as a model for cell differentiation. Large systems of binary variables—representing on and off genes—were studied regarding their development in time, which is determined by the interaction between the elements. The behavior of such genetic nets gives quite a few hints on how a certain order required for living organisms can arise out of complete disorder. The Kauffman model works in the following way: at the beginning each lattice site gets a randomly chosen initial value σ_i which is 1 or 0. The time evolution is determined by a set of Boolean functions which describe mathematically the interaction between the elements of the net. In each time step, spin i is influenced by K other variables of the same system and its new value at time $t+1$ is given by the value of a function

$$f_i: \{0, 1\}^K \to \{0, 1\}$$

for a given configuration of K input spins:

$$\sigma_i(t+1) = f_i(\sigma_{i1}(t), \ldots, \sigma_{iK}(t))$$

[1] Institute of Theoretical Physics, University of Cologne, 5000 Cologne 41, West Germany.
[2] Present address: Merowingerstrasse 54, 5000 Cologne 1, West Germany.

0022-4715/88/1100-0995$06.00/0

K is called the connectivity of the system. The fact that apart from the initial spin configuration also the Boolean functions are chosen at random for each spin caused the great disorder in this kind of cellular automata.

Already Kauffman investigated the stability of such a genetic system against small mutations. The asymptotic damage $\psi(t \to \infty)$ is then the number of sites affected by this mutation after a sufficiently long time. Derrida and Stauffer[4] studied such damage spreading in the square Kauffman model by varying the choice of Boolean functions instead of the connectivity. Here the set of interaction rules is biased by a parameter p, which is the probability for a function at a given neighbor configuration to give the value 1. They observed a phase transition from a frozen phase (for small p), where damage mostly remains limited, into a chaotic phase (for large values of p), where damage is more likely to spread over the whole lattice. At the transition point p_c one can define the proportionalities

$$t \propto L^{D'} \qquad \text{and} \qquad M_{\text{act}} \propto L^{D}$$

where t is the time the damage took to touch the boundary of the system and M is the mass of this damage.

The scaling exponents D' and D are then the fractal dimensions of time and mass, which determine the universality class of this model. Analogously, de Arcangelis[6] and Hansen[9] presented the values of p_c and the critical exponents for the cubic and the four-dimensional lattice. The standard square model was compared with some variations to check for universality of the exponents[7]; sequential updating seemed to change the exponents in these preliminary tests.[7]

The purpose of the present study is to compare the results for simultaneous updating with the behavior of the square lattice where the spins are sequentially updated.

For the classical model with parallel updating the time evolution is realized in the following way: in each time step for every spin its new value is calculated using the specific Boolean function and then it is stored. After this has been done for the sites of the whole lattice, at the end of the time step they are all updated simultaneously. In the sequential model each spin is immediately flipped when its new value has been calculated. This way every lattice sites is influenced by the old values (at time t) of its right and lower neighbors, but also by the new values (at time $t+1$) of its left and upper neighbors, if we go through the lattice like a typewriter. Obviously, the time steps themselves are now divided into N different, smaller time steps and the number of possible states of the whole lattice has increased to $N * 2^N$ from 2^N possible configurations in the classical case.

Furthermore, the damage has the chance to reach the boundaries during the first step and instead of spreading symmetrically into all four

directions it will probably spread faster downward and to the right. So touching time and actual damage are expected to be smaller than with parallel updating.

The simulations have been realized on an IBM 4341 computer with 32 bits per word, which made it possible to store 32 spin values in one computer word using multispin-coding technique.[11,12] Unfortunately, this programming technique based upon the da Silva–Herrmann algorithm is not able to model sequential updating correctly, so partly these experiments had to be realized by a slower program (12.6 μsec/step instead of 1.4 μsec/step with the fast program), using a whole computer word for the storage of one single spin value.

To determine p_c, I used two different methods. First following the Landau theory of phase transitions, I looked for the value of p where the order parameter $\psi(\infty)$ vanishes. At the center of a 64×64 square lattice an initial damage was introduced and after 1000 time steps the Hemming distance was registered and plotted against different values of p. Several tests showed that $\psi(1000)$ is a good approximation of $\psi(\infty)$ for a lattice of that size. The order parameter vanishes at $p_c = 0.29 \pm 0.01$ for both simultaneous and sequential updating, consistent with known results (Fig. 1).[11,12] Similar investigations of lattices initialized with a concentration p instead of 0.5 of spins of the value 1 gave the same results for p_c.

Then I used another method to determine p_c which takes the specific properties of the two phases and the transition point into consideration. In the frozen phase below p_c in most cases the damage should remain finite, i.e., not touch the boundaries of the system. As the lattice itself is of finite size, the success rate of damage spreading not only depends on p, but also on the system size. If the mean damage cloud size exceeds the size of the lattice, nevertheless a large rate of success will be observed in the frozen phase, too. Increasing L, a sudden decrease of successful touching will be noticed as soon as the system gets larger than the average damage cloud.

In contrast, in the chaotic phase the success rate is quite independent of lattice size; the rate of success remains constant at about 50–60 % in the classical model. At $p = p_c$ success rates decrease smoothly with increasing system size. Figure 2 shows the percentage of successful runs (total 1000 runs) plotted against the linear dimension L of the systems for different p. In spite of strong fluctuations, the result of $p_c = 0.29 \pm 0.01$ for both parallel and sequential updating has been confirmed by this second experiment. Another series of simulations was realized for lattices initialized with a concentration p of up spins, but no disagreement of the results has been noticed.

For the investigation of fractal dimensions I simulated systems of size $L = 32$, 64, 96, and 128 at $p = 0.29$ (1000 runs each) both with parallel and

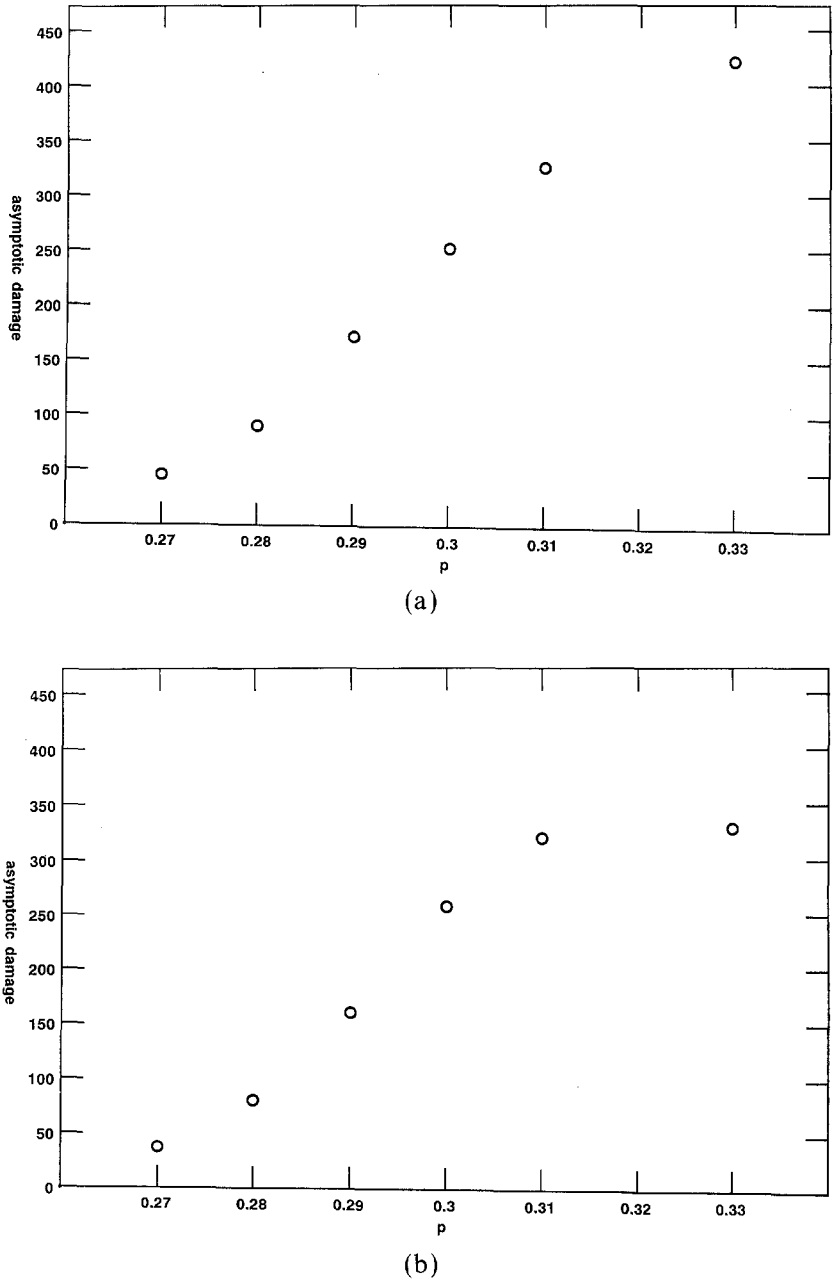
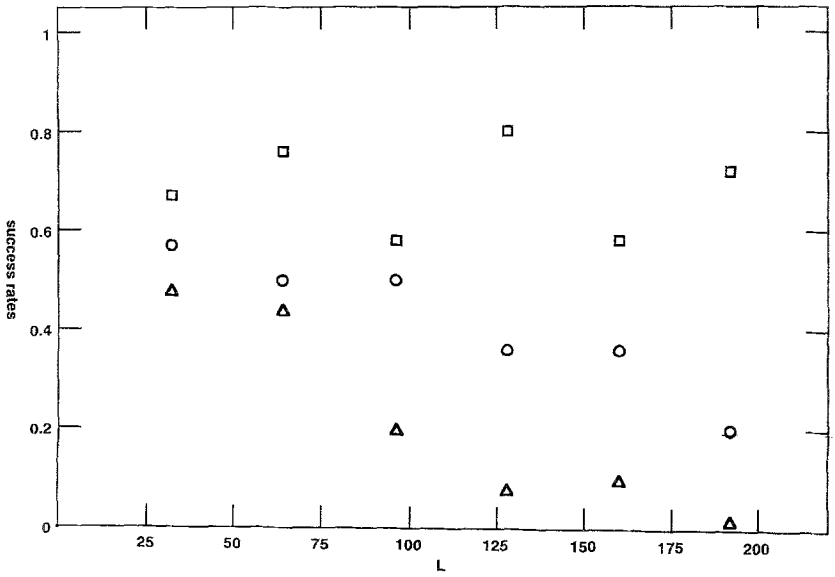
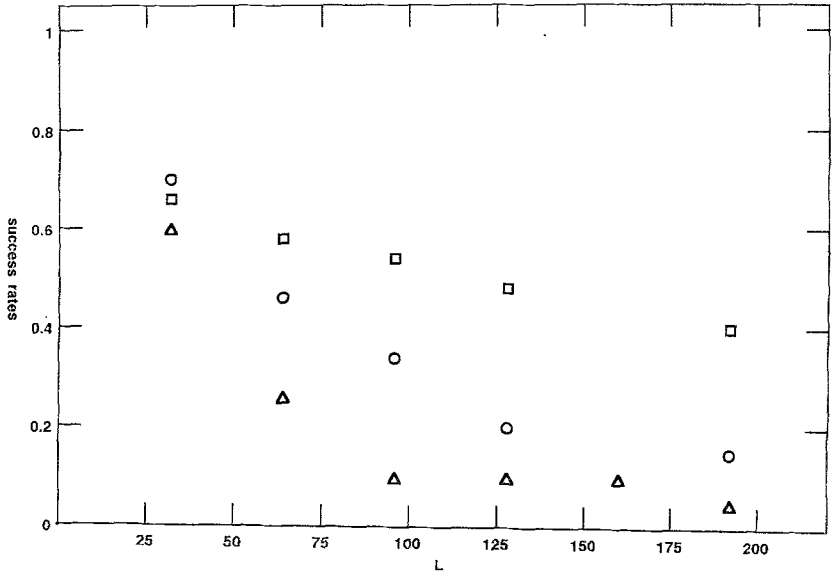


Fig. 1. (a) Damage after 1000 time steps $\psi(\infty)$ versus p in a 64×64 lattice with simultaneous updating. $\psi(\infty)$ vanishes at 0.29 ± 0.01. (b) Same as (a) for sequential updating.

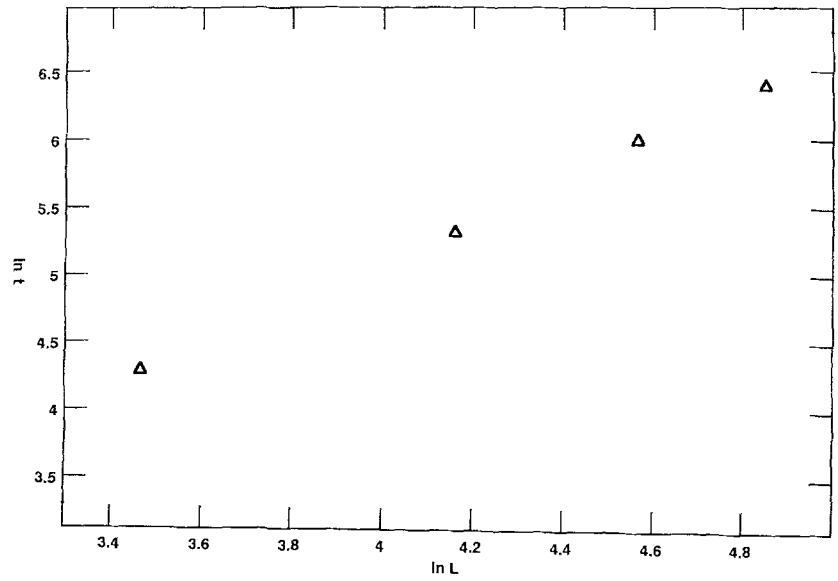


(a)

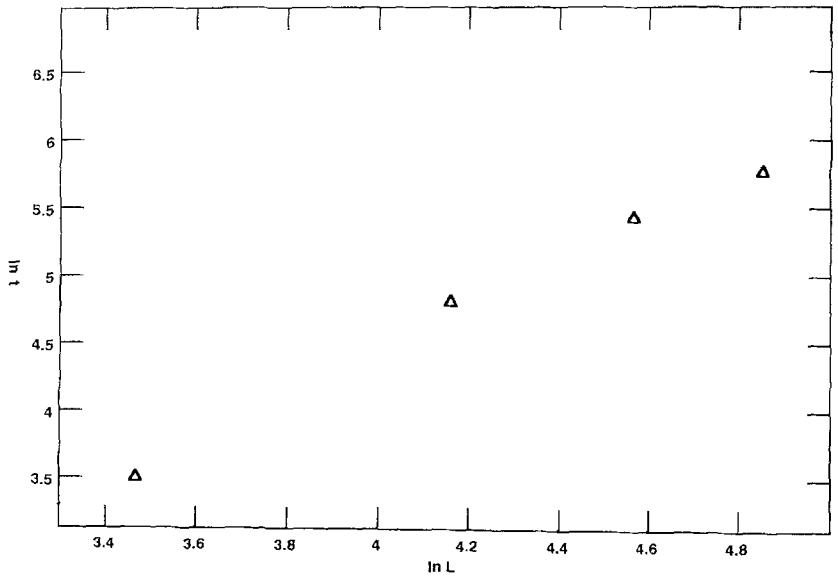


(b)

Fig. 2. (a) Parallel updating: Success rates (total 1000 runs) versus linear dimension L of the system for different values of p. A smooth decrease of success rates is seen for $p = 0.29 \pm 0.01$. (b) Same as (a) for sequential updating. △, $p = 0.28$; ○, $p = 0.29$; □, $p = 0.3$.

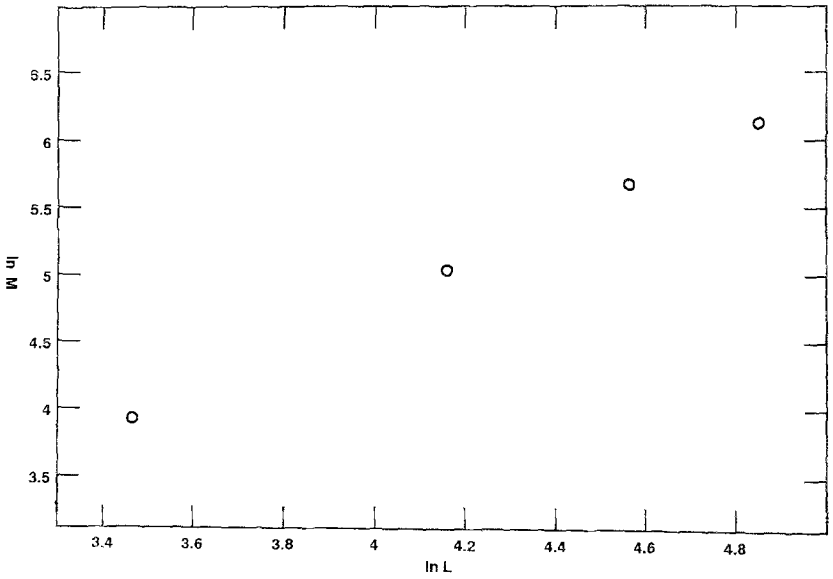


(a)

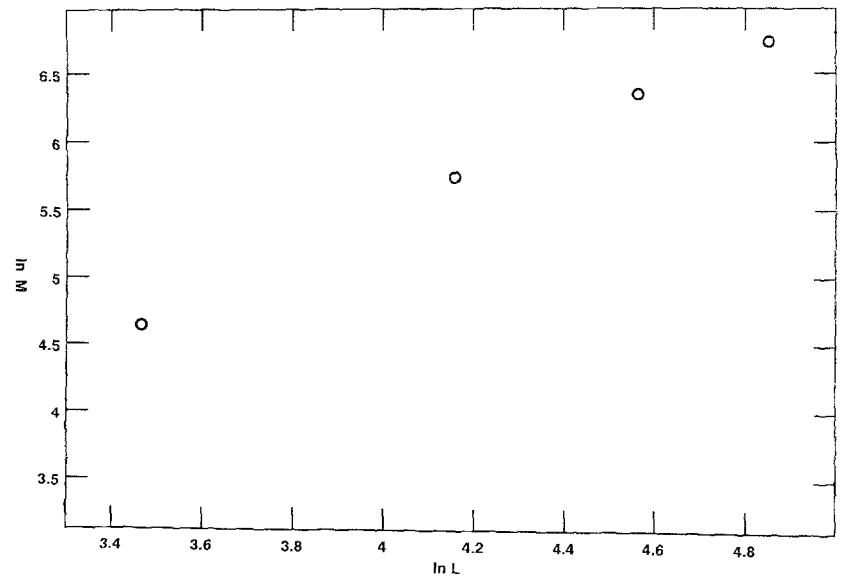


(b)

Fig. 3. (a) The average time for a damage to touch the boundaries of a system versus linear system size L (log–log plot) at $p = 0.29$. The slope is the scaling exponent of time $D'_{\text{par}} = 1.54$ (parallel updating, 1000 runs). (b) Same as (a) for sequential updating slope: $D'_{\text{seq}} = 1.65$.

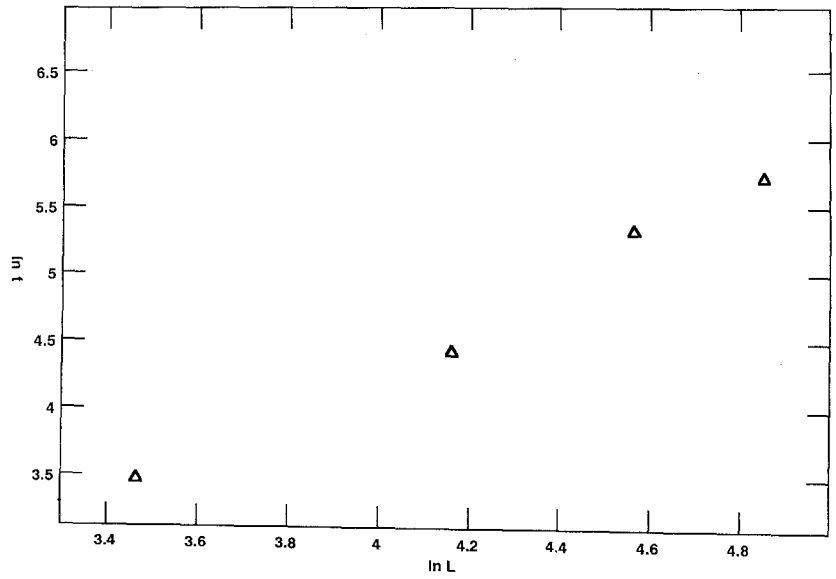


(a)

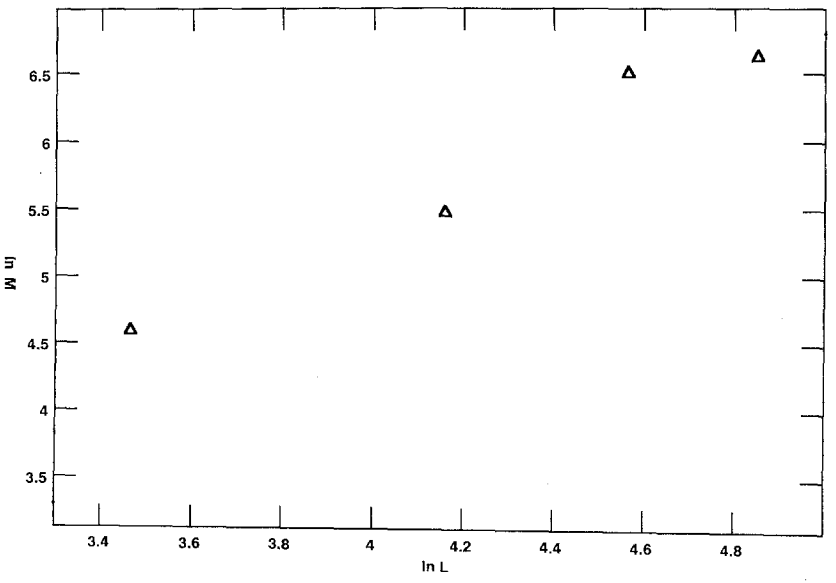


(b)

Fig. 4. (a) The actual damage averaged over the succesful runs plotted versus L at $p = 0.29$ (log–log plot). The slope is the scaling exponent of damage $D_{\text{par}} = 1.59$ (parallel updating, 1000 runs). (b) Same as (a) for sequential updating slope: $D_{\text{seq}} = 1.54$.



(a)



(b)

Fig. 5. Sequential updating simulated with correct boundary conditions (slow algorithm, 200 runs). (a) Average touching time versus L (log–log plot). Slope $D'_{\text{seq}} = 1.65$. (b) Average actual damage versus L (log–log plot). Slope $D_{\text{seq}} = 1.56$.

sequential updating. Touching time and actual damage were averaged over all successful runs. The slope of log–log plots of time and mass against one-dimensional system size L gives the corresponding critical exponent (Figs. 3 and 4):

$$D'_{\text{par}} = 1.54 \pm 0.1, \qquad D_{\text{par}} = 1.59 \pm 0.05$$
$$D'_{\text{seq}} = 1.65 \pm 0.1, \qquad D_{\text{seq}} = 1.54 \pm 0.05$$

(statistical errors only). These simulations were done using multispin coding. It has been mentioned earlier that this technique may not give the correct results for the case of sequential updating, although therein simultaneousness is also violated, but in a slightly biased way. So finally the sequential model was treated again, this time with the help of a slower program without bit operations, to confirm the results obtained before. Figure 5 shows the log–log plots of time and actual damage against L; the values

$$D'_{\text{seq}} = 1.65 \pm 0.2, \qquad D_{\text{seq}} = 1.56 \pm 0.1$$

perfectly agree with the former results.

CONCLUSION

I have investigated the two-dimensional Kauffman model on a square lattice and observed a phase transition for simultaneous as well as for sequential updating of the spins. In both versions the transition occurs at the same value of

$$p_c = 0.29 \pm 0.01$$

The critical exponents of the time an initial damage needs to touch the boundaries of the system and the mass of this damage have been measured and have turned out to be the same for both models,

$$D' = 1.6 = 0.2, \qquad D = 1.6 \pm 0.1$$

in agreement with the results for the standard square model.[7,13]

ACKNOWLEDGMENT

I thank D. Stauffer for suggesting the subject of this work and for many helpful discussions.

NOTE ADDED IN PROOF

Similar results were found for the Ising model by A. U. Neumann and B. Derrida, *J. Phys. (Paris)* **49**:1647 (1988) and for Q2R cellular automata by U. Maeks, unpublished.

REFERENCES

1. S. A. Kauffman, *J. Theor. Biol.* **22**:437–467 (1969).
2. B. Derrida and Y. Pomeau, *Europhys. Lett.* **1**:45–49 (1986).
3. B. Derrida and G. Weisbuch, *J. Phys.* (Paris) **48**:1297–1303 (1986).
4. B. Derrida and D. Stauffer, *Europhys. Lett.* **2**:739–754 (1986).
5. G. Weisbuch and D. Stauffer, *J. Phys.* (Paris) **48**:11–18 (1987).
6. L. De Arcangelis, *J. Phys. A Math. Gen.* **20**:L396–L373 (1987).
7. D. Stauffer, *Phil. Mag. B* **56**:901–916 (1987).
8. D. Stauffer, in *Computer Simulation and Computer Algebra* (Springer, 1988).
9. A. Hansen, *J. Phys. A Mat. Gen.* **21**:2481 (1988).
10. L. R. Da Silva, *J. Stat. Phys.*, to appear.
11. L. R. Da Silva and H. J. Herrmann, *J. Stat. Phys.* **52**:463 (1988).
12. H. J. Herrmann, *J. Stat. Phys.* **45**:145 (1986).
13. D. Stauffer, private communication.

Communicated by D. Stauffer